INTEGRAL RESTRICTIONS ON THE MONODROMY

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Abstract. Given a complex analytic function with a one-dimensional critical locus at the origin, we examine the monodromy action on the integral cohomology of the Milnor fiber. We relate this monodromy to that of a generic hyperplane slice through the origin, and to that of a generic hyperplane slice near the origin. We thereby obtain number-theoretic restrictions on the monodromy and on the cohomology of the original Milnor fiber.

§1. **Introduction**

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and let z_0 be a non-zero linear form on $\mathbb{C}^{n+1}$. Let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function with a 1-dimensional critical locus at the origin, i.e., $\dim_{\mathbf{0}}\Sigma f = 1$. We assume that $\mathcal{U}$ is chosen small enough so that $\Sigma f \subseteq V(f)$.

Let $H_0 := \mathcal{U} \cap V(z_0)$ and let $H_t := \mathcal{U} \cap V(z_0 - t)$, where $t \neq 0$ is sufficiently small. Let $f_0 := f_{|_{H_0}}$ and $f_t := f_{|_{H_t}}$. We assume that z_0 is generic enough so that $\Sigma(f_0) = \{\mathbf{0}\}$. This implies that $\Sigma(f_t)$ consists of a finite number of points: the points of $H_t \cap \Sigma f$, and the points where H_t intersects the relative polar curve Γ^1_{f,z_0}.

Let F denote the Milnor fiber of f at $\mathbf{0}$, and let $m_{n-1} : \widetilde{H}^{n-1}(F;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $m_n : \widetilde{H}^{n}(F;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n}(F;\ \mathbb{Z})$ denote the corresponding f-monodromy actions on reduced integral cohomology. Note that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is free-Abelian. Let F_0 denote the Milnor fiber of f_0 at $\mathbf{0}$, and let

$$h_0 : \mathbb{Z}^{\mu_0} \cong \widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_0}$$

denote the corresponding f_0-monodromy action, where μ_0 denotes the Milnor number of the isolated critical point. For each point $\mathbf{p}_i \in H_t \cap \Sigma f$, there is an associated Milnor fiber F_i of f_t at $\mathbf{p}_i$, together with the associated f_t-monodromy action on

$$h_i : \mathbb{Z}^{\mu_i} \cong \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_i}.$$

The monodromies of f restricted to H_0 and H_t are compatible with the monodromy of f itself; they are all determined by letting the value of f move in a small circle around the origin in $\mathbb{C}$. In categorical terms, this compatibility is a result of the fact that the monodromy is a natural automorphism of the vanishing cycle functor ϕ_f.

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This compatibility enables us to produce a diagram on which the monodromy acts

$$\begin{array}{ccccc} & & \bigoplus_i \mathbb{Z}^{\mu_i} & & \\ & & \big\downarrow{\scriptstyle\alpha} & & \\ \mathbb{Z}^{\mu_0} & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & \mathbb{Z}^{\lambda^0_f} \\ & & \big\downarrow{\scriptstyle\beta} & & \\ & & \mathbb{Z}^{\omega}, & & \end{array}$$

where both the row and the column are short exact (we have omitted the zeroes on each end), $\lambda^0_f = \left(\Gamma^1_{f,z_0} \cdot V\left(\dfrac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}$ is the 0-th Lê number of f (with respect to z_0 at the origin), and $\omega := \left(\Gamma^1_{f,z_0} \cdot V(f)\right)_{\mathbf{0}}$. It is important, and well-known, that $\omega \geqslant \lambda^0_f$, with equality if and only if $\omega = \lambda^0_f = 0$.

It is also important to note that, by the result of A'Campo [**A**], the trace of of the monodromy maps h_0 and h_i are all equal to $(-1)^n$.

We refer to the N in the middle of the diagram as the *nexus*, and refer to the whole diagram as the *nexus diagram*. In fact, it is fairly unimportant what the nexus of the diagram is; what is important is that there exists such a diagram on which the f-monodromy acts.

Let us denote the pull-back via α and γ by P, and the push-forward via β and δ by Q. Then, it is trivial to show that

$$P \;\cong\; \ker(\beta\circ\gamma) \;\cong\; \ker(\delta\circ\alpha)$$

and

$$Q \;\cong\; \operatorname{coker}(\beta\circ\gamma) \;\cong\; \operatorname{coker}(\delta\circ\alpha)$$

This is important because $\beta\circ\gamma$ is the map induced on cohomology by the Morse-theoretic attaching map given by Lê in [**L1**]. This implies that $\ker(\beta\circ\gamma) \cong \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $\operatorname{coker}(\beta\circ\gamma) \cong \widetilde{H}^{n}(F;\ \mathbb{Z})$. An alternate way of seeing these isomorphisms is in terms of Lê numbers (see [**Ma2**]); the $\mathbb{Z}$-module $\bigoplus_i \mathbb{Z}^{\mu_i}$ is precisely $\mathbb{Z}^{\lambda^1_f}$, where λ^1_f is the 1-dimensional Lê number. The map $\delta\circ\alpha : \mathbb{Z}^{\lambda^1_f} \to \mathbb{Z}^{\lambda^0_f}$ is precisely the integral version of the one non-trivial map appearing in the chain complex of Corollary 10.10 of [**Ma2**] (though there is a typographically error in this corollary – the arrows are reversed). As the cohomology of this complex yields the reduced cohomology of F, we again conclude that $P \cong \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $Q \cong \widetilde{H}^{n}(F;\ \mathbb{Z})$.

However, it is not only the f-monodromy that acts on the nexus diagram. We shall see that the z_0-monodromy also acts commutatively on the diagram, acting as the identity on the $\mathbb{Z}^{\mu_0}$ and $\mathbb{Z}^{\omega}$ nodes. Moreover, since the z_0 monodromy is also a natural isomorphism, it follows that the actions of the f-monodromy and z_0-monodromy on the nexus diagram commute with each other.

To continue our discussion, we must adopt some notation for the z_0-monodromy action on $\bigoplus_i \mathbb{Z}^{\mu_i}$. Let ν be a (reduced) component of Σf. Assume that $\mathcal{U}$ is small enough so that ν is homeomorphic to a disk, and small enough so that the vanishing cycles along f, restricted to the punctured disk $\nu^* := \nu - \{\mathbf{0}\}$, form a local system. Then, there is an *internal* (or *vertical*) monodromy action ι_ν induced on a given stalk of the vanishing cycle local system; one moves once around the "hole" in the punctured disk ν^*.

Now, for each component ν of Σf, there are $k_\nu := (\nu \cdot V(z_0))_{\mathbf{0}}$ points $\mathbf{p}_i$ which occur in $\nu \cap H_t$. At each of these $\mathbf{p}_i$, f_t has the same Milnor number; let us denote this common value by μ_ν. There is a "fractional monodromy" action, $\tau_\nu : \mathbb{Z}^{\mu_\nu} \xrightarrow{\cong} \mathbb{Z}^{\mu_\nu}$ given by moving cyclicly from one $\mathbf{p}_i$ in $\nu \cap H_t$ to the next. It follows that $\tau_\nu^{k_\nu} = \iota_v$. Moreover, if we let λ_ν denote the z_0-monodromy action on $\bigoplus_{\mathbf{p}_i \in \nu \cap H_t} \mathbb{Z}^{\mu_i}$, then it follows that $\lambda_\nu : (\mathbb{Z}^{\mu_\nu})^{k_\nu} \to (\mathbb{Z}^{\mu_\nu})^{k_\nu}$ is given by

$$\lambda_\nu(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k_\nu}) = (\tau_\nu(\mathbf{v}_{k_\nu}), \tau_\nu(\mathbf{v}_1), \ldots, \tau_\nu(\mathbf{v}_{k_\nu - 1})),$$

from which one concludes immediately that $\ker(\mathrm{id} - \iota_\nu) \cong \ker(\mathrm{id} - \lambda_\nu)$.

Returning to the nexus diagram, since the z_0-monodromy acts as the identity on $\mathbb{Z}^{\mu_0}$, it follows that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ must be contained in $\bigoplus_\nu \ker(\mathrm{id} - \lambda_\nu) \cong \bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$. In particular, the rank of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is at most $\sum_\nu \mu_\nu$.

Let us use $\mathrm{char}_{\_}(t)$ to denote characteristic polynomials. Then, using our discussion above and the proof in Section 2, what we show is:

Main Theorem. *The nexus diagram exists, and the f-monodromy and z_0-monodromy act commutatively on it, and these actions commute with each other.*

The z_0-mondromy acts as the identity on $\mathbb{Z}^{\mu_0}$ and on $\mathbb{Z}^{\omega}$. The kernel of the identity minus the z_0-monodromy on $\bigoplus_i \mathbb{Z}^{\mu_i}$ is isomorphic to $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$.

Therefore, $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ injects into $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$. Moreover, if h_ν denotes one of the h_i for $\mathbf{p}_i \in \nu$, then $\mathrm{char}_{m_{n-1}}(t)$ divides $\mathrm{char}_{h_0}(t)$ and $\prod_\nu \mathrm{char}_{h_\nu}(t)$ in $\mathbb{Z}[t]$, i.e.,

$$\mathrm{char}_{m_{n-1}}(t) \ \Big|\ \gcd\Big(\mathrm{char}_{h_0}(t),\ \prod_\nu \mathrm{char}_{h_\nu}(t)\Big).$$

Aside from proving the Main Theorem, we also give a few applications of it. We show how the nexus diagram simplifies the proof of the non-splitting result of Lê in [**L2**]. We show how the Main Theorem allows us to generalize Lê's non-splitting result and obtain the main result of [**Ma1**]. Finally, we show how the Main Theorem generalizes the main result of [**Ma3**] from the case of arrangements of planes in $\mathbb{C}^3$ to the case of arbitrary affine hypersurfaces with 1-dimensional critical loci.

In the final section of the paper, we make some brief remarks which may lead to future applications.

§2. **Proof of the Main Theorem**.

We continue with the notation from the introduction.

Let $k : \mathcal{U} \cap V(z_0) \hookrightarrow \mathcal{U}$, $\hat{k} : V(f) \cap V(z_0) \hookrightarrow V(f)$, and $l : \mathcal{U} - \mathcal{U} \cap V(z_0) \hookrightarrow \mathcal{U}$ denote the inclusions. Let . Let $\hat{z}_0 := z_{0|_{V(f)}}$.

Let $\mathbf{P}^\bullet$ denote the complex of sheaves of $\mathbb{Z}$-modules $\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$; this sheaf is perverse. There is a fundamental distinguished triangle

$$l_!l^!\mathbf{P}^\bullet \to \mathbf{P}^\bullet \to k_*k^*\mathbf{P}^\bullet \xrightarrow{[1]},$$

which we can "turn" to yield

$$k_*k^*\mathbf{P}^\bullet[-1] \to l_!l^!\mathbf{P}^\bullet \to \mathbf{P}^\bullet \xrightarrow{[1]} .$$

Applying the composed functor $\phi_{\hat{z}_0}[-1]\phi_f[-1]$ to the above triangle, we obtain the distinguished triangle

$$(\dagger) \qquad \phi_{\hat{z}_0}[-1]\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \to \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet \xrightarrow{[1]} .$$

Now,

- $k^*\mathbf{P}^\bullet[-1] \cong \mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]$, and so $\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \cong \hat{k}_*\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]$. As the support of this last complex of sheaves is contained in $V(z_0)$, if we apply $\psi_{\hat{z}_0}[-1]$, we get the zero complex. Hence, we find that the first complex in ($\dagger$)

$$\phi_{\hat{z}_0}[-1]\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \cong \phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n].$$

Note that this is a perverse sheaf with isolated support at the origin.

- As $\mathbf{P}^\bullet$ is perverse and l is the inclusion of a hypersurface complement, $l_!l^!\mathbf{P}^\bullet$ is perverse. Thus, the second complex of ($\dagger$), $\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet$, is perverse.

- As $\mathbf{P}^\bullet$ is perverse and the origin is an isolated point in the intersection of $V(z_0)$ and Σf, $\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet$ is perverse and has the origin as an isolated point in its support.

At isolated points in their supports, perverse sheaves have their stalk cohomology concentrated in degree zero. Thus, the long-exact sequence on the stalk cohomology at the origin obtained from ($\dagger$) has at most one non-trivial piece – namely, the short exact sequence

$$(\ddagger) \;\; 0 \to H^0\big(\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]\big)_{\mathbf{0}} \to H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}} \to H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet\big)_{\mathbf{0}} \to 0.$$

Observe that $H^0\big(\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]\big)_{\mathbf{0}} \cong \mathbb{Z}^{\mu_0}$ and that, by [**Ma2**] or [**Ma4**],

$$H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big)_{\mathbf{0}} \cong \mathbb{Z}^{\lambda^0_f}.$$

Therefore, we define the nexus N to be $H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}}$, and ($\ddagger$) becomes the horizontal row of the nexus diagram.

To obtain the vertical row of the nexus diagram, we begin by considering another fundamental distinguished triangle

$$\hat{k}^*[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]}$$

and turn this triangle to obtain

$$\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \hat{k}^*\phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]} .$$

Again, we look at the associated long exact sequence on the stalk cohomology at the origin.

- As $l_!l^!\mathbf{P}^\bullet$ is isomorphic to $\mathbf{P}^\bullet$ outside of $V(z_0)$, it follows that

$$\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \;\cong\; \psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet.$$

Since $\mathbf{P}^\bullet$ is perverse and the origin is an isolated point in the intersection of $V(z_0)$ and Σf, $\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet$ is perverse and has the origin as an isolated point in its support. Thus,

$$H^0\big(\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}} \;\cong\; H^0\big(\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet\big)_{\mathbf{0}} \;\cong\; \bigoplus_i \mathbb{Z}^{\mu_i}.$$

- Note that $H^*\big(\hat{k}^*\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0 \;\cong\; H^*\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0$ As the nexus is defined to be $N = H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}}$, we would be finished if we could show that $H^*\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0$ is zero outside of degree zero and that

$$H^0\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0 \cong \mathbb{Z}^\omega.$$

Consider the distinguished triangle

$$\big(l_!l^!\mathbf{P}^\bullet\big)_{|_{V(f)}} \to \psi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]} .$$

As $l_!$ is the extension by zero, we find that

$$H^0\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0 \cong H^0\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0.$$

Now, consider the distinguished triangle

$$\psi_f[-1]l_!l^!\mathbf{P}^\bullet \to \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\xi} \psi_f[-1]k_*k^*\mathbf{P}^\bullet \xrightarrow{[1]} .$$

The stalk at $\mathbf{0}$ of the map ξ is the map induced by Lê's attaching map in [**L1**]; this is Morse-theoretic map involved in Lê's description of how the Milnor fiber of f is obtained from the Milnor fiber of f_0. Therefore, $H^*\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0$ is isomorphic to the relative cohomology $H^*(F, F_0;\ \mathbb{Z})$, and Lê's result tells us that $H^i\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0 = 0$, unless $i = 0$, and

$$H^0\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_0 \cong \mathbb{Z}^\omega.$$

This is what we needed to prove.

§3. **Applications**.

Application 1. We will use the Main Theorem to simplify the non-splitting result of Lê which appears in [**L2**]. The heart of the argument is same; it is the details and "trick" at the end of Lê's proof that we can eliminate.

Suppose that $\mu_0 = \lambda^1_f$, i.e., that $\mu_0 = \sum_i \mu_i$. We wish to conclude that there is exactly one point in $H_t \cap \Sigma f$, i.e., that the intersection number of reduced varieties $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}$ equals one. This is equivalent to saying that Σf has a single smooth component which is transversely intersected by $V(z_0)$.

As $\mu_0 = \lambda^1_f$, from the nexus diagram we conclude that $\omega = \lambda^0_f$. However, as discussed in the Introduction, this implies that $\omega = \lambda^0_f = 0$. Thus, α and γ are isomorphisms. As the f-monodromy maps act commutatively on the nexus diagram, the result of A'Campo implies that

$$(-1)^n = \big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}(-1)^n,$$

and hence $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} = 1$.

Note that the nexus diagram also implies that, in this case, $\widetilde{H}^n(F;\ \mathbb{Z}) = 0$ and $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \widetilde{H}^{n-1}(F_0;\ \mathbb{Z})$.

Application 2. The fact that $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \ker(\delta \circ \alpha)$ immediately implies that the rank of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is at most λ^1_f. Assume that $\operatorname{rk} \widetilde{H}^{n-1}(F;\ \mathbb{Z}) = \lambda^1_f$; we wish to see what this implies.

Recall from the Introduction that $\operatorname{rk} \widetilde{H}^{n-1}(F;\ \mathbb{Z}) \leqslant \sum_\nu \mu_\nu$. As $\lambda^1_f = \sum_\nu (\nu \cdot V(z_0))_{\mathbf{0}}\, \mu_\nu$, we see that we must have that, for every component ν of Σf, $(\nu \cdot V(z_0))_{\mathbf{0}} = 1$, i.e., each ν is smooth at the origin and is transversely intersected by $V(z_0)$.

Now, $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \mathbb{Z}^{\lambda^1_f}$ injects into the copy of $\mathbb{Z}^{\lambda^1_f}$ in the nexus diagram. Hence, the characteristic polynomials of the f-monodromy on $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and on the copy of $\mathbb{Z}^{\lambda^1_f}$ in the nexus diagram must be equal. Using that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ injects into $\widetilde{H}^{n-1}(F_0;\ \mathbb{Z})$ and applying the result of A'Campo, we conclude that $\widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_0}$ has an f-monodromy-invariant free submodule of rank λ^1_f on which the f-monodromy acts with trace $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}(-1)^n$. Since the eigenvalues of the f-monodromy are all roots of unity, and the trace of h_0 is itself equal to $(-1)^n$, we find that

$$\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}(-1)^n + \big(\text{the sum of } \mu_0 - \lambda^1_f \text{ roots of unity}\big) = (-1)^n,$$

or

$$(\dagger) \qquad \Big(\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} - 1\Big)(-1)^n \ = \ \text{the sum of } \mu_0 - \lambda^1_f \text{ roots of unity.}$$

Taking norms, we conclude that

$$(\ddagger) \qquad \big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} - 1 \leqslant \mu_0 - \lambda^1_f$$

with equality implying that h_0 has $\mu_0 - \lambda^1_f$ eigenvalues equal to $(-1)^{n+1}$.

- The case in Application 1 was that of $\mu_0 - \lambda^1_f = 0$. We see that from ($\ddagger$), we must have $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} = 1$ in this case.

- Consider now the next easiest case where $\mu_0 - \lambda^1_f = 1$. Then, ($\ddagger$) implies that $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}$ is equal to 1 or 2. However, ($\dagger$) tells us that we cannot have $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} = 1$.

Therefore, we must have $\big(|\Sigma f|\cdot V(z_0)\big)_{\mathbf{0}}=2$. This implies that Σf has two smooth components which are each transversely intersected by $V(z_0)$.

By taking the contrapositive of the above work, we recover a slightly-improved version of the result of [**Ma1**]: if $\mu_0-\lambda^1_f=1$, then either Σf consists of two smooth components which are transversely intersected by $V(z_0)$ at the origin, or the rank of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is strictly less than λ^1_f.

Application 3. Suppose that f_0 is homogeneous of degree d_0. By using Theorem 9.6 of [**Mi**], or directly from [**M-O**], the characteristic polynomial of the f_0-monodromy is given by

$$(\dagger)\qquad \operatorname{char}_{h_0}(t)=(t-1)^{a_0}\left(\frac{t^{d_0}-1}{t-1}\right)^{b_0},$$

where $b_0:=\dfrac{(d_0-1)^n-(-1)^n}{d_0}$ and $a_0:=b_0+(-1)^n=(d_0-1)\left(\dfrac{(d_0-1)^{n-1}-(-1)^{n-1}}{d_0}\right)$.

Let Φ_k denote the k-th cyclotomic polynomial, i.e., $\Phi_k=\prod_\xi(t-\xi)$ where the ξ vary over the primitive k-th roots of unity. Then, the main theorem immediately tells us that

$$\operatorname{char}_{m_{n-1}}(t)=\prod_{k|d_0}\Phi_k^{c_k},$$

where $c_1\leqslant a_0$ and, for $k>1$, $c_k\leqslant b_0$; in addition, for all $k|d_0$, we must have that $\Phi_k^{c_k}$ divides $\prod_\nu \operatorname{char}_{h_\nu}(t)$ in $\mathbb{Z}[t]$.

This is particularly useful in the special case where, at each point $\mathbf{p}_i\in H_t\cap\Sigma f$, f_t is "homogeneous at $\mathbf{p}_i$", i.e., $g_i(\mathbf{z}):=f_t(\mathbf{z}+\mathbf{p}_i)$ is homogeneous. Let d_i denote the degree of g_i. For all $\mathbf{p}_i$ in a given component ν of Σf, the d_i must be the same; denote this common value by d_ν. For each ν, there is a formula analogous to ($\dagger$) for the characteristic polynomial of h_ν, and so the irreducible factors of $\prod_\nu \operatorname{char}_{h_\nu}(t)$ are cyclotomic polynomials Φ_k for which k must divide one of the d_ν.

An example of such an f would be one which defines a central arrangement of d_0 hyperplanes in $\mathbb{C}^3$. The above paragraphs generalize most of the main theorem from [**Ma3**], where we considered only hyperplane arrangements in $\mathbb{C}^3$. This is related to the work of [**E**].

§4. **Further Remarks**.

Regardless of the dimension of the critical locus of f, a nexus diagram exists in the category of perverse sheaves.

In the proof of the Main Theorem, even before we took the stalk cohomology at the origin, we had two short exact sequences in the category of perverse sheaves; that is, we had a perverse nexus diagram. However, since each of these perverse sheaves had the origin as an isolated point in their support, taking stalk cohomology essentially did nothing.

If the dimension of Σf is arbitrary and z_0 is generic, we still obtain a perverse nexus diagram

$$\begin{tikzcd}
 & \psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \arrow[d, "\alpha"] & \\
\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n] \arrow[r, "\gamma"] & \mathbf{N}^\bullet \arrow[r, "\delta"] \arrow[d, "\beta"] & j_*\mathbb{Z}_{\mathbf{0}}^{\lambda_f^0} \\
 & j_*\mathbb{Z}_{\mathbf{0}}^{\omega}, &
\end{tikzcd}$$

where j denotes the inclusion of the origin into $V(f, z_0)$. The f and z_0 monodromies act commutatively on this diagram.

We write $\ker^p$ and coker^p for the kernel and cokernel in the category of perverse sheaves, and we write ${}^pH^*$ for the perverse cohomology. Then, in general, in the perverse nexus diagram above,

$$\ker^p(\beta\circ\gamma) \;\cong\; \ker^p(\delta\circ\alpha) \;\cong\; {}^pH^{-1}\big(\hat{k}^*\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big)$$

and

$$\operatorname{coker}^p(\beta\circ\gamma) \;\cong\; \operatorname{coker}^p(\delta\circ\alpha) \;\cong\; {}^pH^{0}\big(\hat{k}^*\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big).$$

Furthermore, there is an analogous perverse nexus diagram involving $\psi_f[-1]$, in place of $\phi_f[-1]$. That is, there is a nexus diagram in the category of perverse sheaves obtained from the above diagram by replacing each occurrence of $\phi_f[-1]$ by $\psi_f[-1]$, and by replacing λ_f^0 by ω. The perverse kernel and cokernel statements also hold with $\phi_f[-1]$ replaced by $\psi_f[-1]$.

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